Endpoint Relations on Temporal Intervals

David Matuszek, Tim Finin, Richard Fritzson, Christian Overton

Unisys Corporation
Paoli Research Center
P. O. Box 517
Paoli, PA 19301

Abstract

Allen's interval algebra has become one of the major schemes used in AI for temporal reasoning. However, Allen's algebra has the esthetically displeasing property that the less one knows about the relationship between two intervals, the longer must be the symbolic expression of that knowledge. This paper presents a reformulation of Allen's interval algebra in terms of more primitive relations between intervals, such that less knowledge requires a shorter symbolic expression. In addition, the reformulation results in a significant simplification of Allen's transitivity table.

Introduction.

Much of the contemporary AI research in temporal reasoning is based on Allen's work [Allen 83; Allen 84; Allen 85]. Allen's interval algebra is concerned with the relationships between time intervals, rather than time points. Such a focus enables a better description of processes and events.

Allen noted that there were thirteen possible relationships between two intervals (equality is its own inverse relationship). These are summarized in Figure 1, along with Allen's symbols and names for the relationships and inverse relationships. Intervals are required to be of nonzero duration.

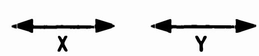
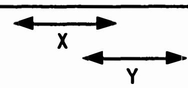
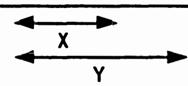
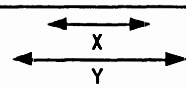
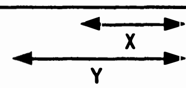
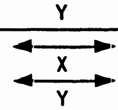
Symbol	Picture	English phrase
X < Y, Y > X		X before Y Y after X
X m Y, Y mi X		X meets Y Y met-by X
X o Y, Y oi X		X overlaps Y Y overlapped-by X
X s Y, Y si X		X starts Y Y started-by X
X d Y, Y di X		X during Y Y contains X
X f Y, Y fi X		X finishes Y Y finished-by X
X = Y		X equal Y

Figure 1. Allen's temporal relations.

The current work grew from misgivings about the "naturalness" of Allen's system. Two problems in particular can be noted. First, when the relationship between two intervals is known completely, it can be represented by a brief expression in the interval algebra. For example, if it is known that intervals A and B start at the same instant, but that interval A ends before interval B does, this can be represented as (A s B). However, if less is known, the symbolic expression becomes longer. For example, if it is known only that intervals A and B start at the same time, this must be represented as (A s B) v (A = B) v (A si B). This "less is more" representation seems counterintuitive.

Second, Allen's interval algebra makes heavy use of disjunction. There is considerable psychological evidence [Neisser 62; Nitta 66; Neimark 70; Salatas 74; Matuszek 78] that disjunction in any form causes difficulties for humans; see especially [Neimark 70]. It is outside the scope of this paper to defend this point; however, it may be noted that recent work indicates that Allen's use of disjunctions may cause some formal problems as well [Tsang 87; Valdes-Perez 87].

A Revised Set of Temporal Relations

Each of Allen's thirteen possible relations describes *in full* the relationship between two intervals. If the relationship is not fully known, then one must give a set of relations, in effect saying "the relationship between the two intervals must be one of the fully specified relations in this set."

When partial knowledge about the relationship of two intervals is available, it is usually in the form of information about the endpoints of those intervals. By concentrating on the start and finish points of intervals, we have devised an alternate set of more primitive relations between intervals, as shown in Figure 2. For ease of reference, we shall call these relations *endpoint relations.*

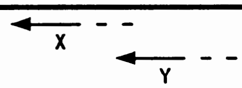
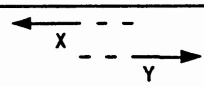
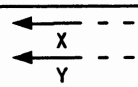
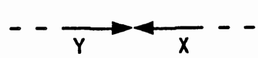
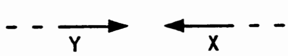
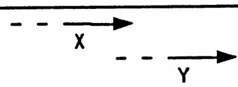
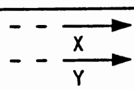
Symbol	Picture	English phrase
X sbs Y Y sas X		X starts before Y starts Y starts after X starts
X sbf Y Y fas X		X starts before Y finishes Y finishes after X starts
X sws Y Y sws X		X starts when Y starts Y starts when X starts
X swf Y Y fws X		X starts when Y finishes Y finishes when X starts
X saf Y Y fbs X		X starts after Y finishes Y finishes before X starts
X fbf Y Y faf X		X finishes before Y finishes Y finishes after X finishes
X fwf Y Y fwf X		X finishes when Y finishes Y finishes when X finishes

Figure 2. Endpoint relations on intervals.

It should be emphasized that this set of relations does not constitute a retreat to a point-based temporal logic. Endpoint relations are still relations between *intervals* rather than points, and as such are simply a repackaging of Allen's relations. Either set of relations can be defined in terms of the other set, as shown in Figures 3 and 4.

X sbs Y	(X < Y) or (X m Y) or (X o Y) or (X fi Y) or (X di Y)
X sbf Y	(X < Y) or (X m Y) or (X o Y) or (X fi Y) or (X di Y) or (X s Y) or (X = Y) or (X si Y) or (X d Y) or (X f Y) or (X oi Y)
X sws Y	(X s Y) or (X = Y) or (X si Y)
X swf Y	X mi Y
X sas Y	(X d Y) or (X f Y) or (X oi Y) or (X mi Y) or (X > Y)
X saf Y	X > Y
X fbs Y	X < Y
X fbf Y	(X < Y) or (X m Y) or (X o Y) or (X s Y) or (X d Y)
X fws Y	X m Y
X fwf Y	(X fi Y) or (X = Y) or (X f Y)
X fas Y	(X o Y) or (X fi Y) or (X di Y) or (X s Y) or (X = Y) or (X si Y) or (X d Y) or (X f Y) or (X oi Y) or (X mi Y) or (X > Y)
X faf Y	(X di Y) or (X si Y) or (X oi Y) or (X mi Y) or (X > Y)

Figure 3. Endpoint relations defined in terms of Allen's relations.

X < Y	X fbs Y
X m Y	X fws Y
X o Y	(X sbs Y) and (X fbf Y) and (X fas Y)
X fi Y	(X sbs Y) and (X fwf Y)
X di Y	(X sbs Y) and (X faf Y)
X s Y	(X sws Y) and (X fbf Y)
X = Y	(X sws Y) and (X fwf Y)
X si Y	(X sws Y) and (X faf Y)
X d Y	(X sas Y) and (X fbf Y)
X f Y	(X sas Y) and (X fwf Y)
X oi Y	(X sas Y) and (X fbf Y) and (X sbf Y)
X mi Y	X swf Y
X > Y	X saf Y

Figure 4. Allen's relations defined in terms of endpoint relations.

The tradeoff between these two sets of relations is that, generally speaking, Allen's relations are more succinct when the relationship between two intervals is completely specified; the endpoint relations are more succinct when the relationship is incompletely specified. It can be seen from the tables that each of Allen's relations may be defined in terms of from one to three endpoint relations (average, 1.8), and each of the endpoint relations may be defined in terms of from one to eleven of Allen's relations (average, 4.3).

Composite relationships seem to be simplified correspondingly. For example, to express in Allen's system that X and Y are contemporaries (i.e. they have some nonzero subinterval in common), we would write

```
(X o Y) or (X oi Y) or (X s Y) or (X si Y) or (X d Y) or (X di Y)
     or (X f Y) or (X fi Y) or (X = Y).
```

This can be shortened somewhat by introducing negation:

```
not((X < Y) or (X > Y) or (X m Y) or (X mi Y)).
```

Using endpoint relations, the "contemporaries" relationship is simply

```
(X sbf Y) and (X fas Y).
```

Vilain and Kautz [Vilain 86] note that a system based on endpoints cannot represent all possible relationships between intervals unless one allows disjunction as well as conjunction. For example (X < Y) v (X > Y) must be represented by (X fbs Y) v (X saf Y); it cannot be represented with conjunction alone. However, where conjunction is sufficient, a more efficient *algebra of time points* (rather than intervals) may be employed.

Transitivity tables.

More important than the ease of expressing the relationships, however, is the ease of computing with them. An important basic computation is as follows: Given the relation between intervals X and Y, and the relation between intervals Y and Z, determine the relation between intervals X and Z. Allen refers to this as *transitivity*, and provides a transitivity table (see Figure 5) for pairs of relations: If row *i* expresses the relationship between intervals X and Y, and column *j* specifies the relationship between intervals Y and Z, then the entry at table location ⟨*i*, *j*⟩ specifies the possible relationships between X and Z. For example, if X meets Y, and Y is during Z, then X either is during, starts, or overlaps Z.

	=	<	>	d	di	o	oi	m	mi	s	si	f	fi
"equal" =	=	<	>	d	di	o	oi	m	mi	s	si	f	fi
"before" <	<	<	-	< o m d s	<	<	< o m d s	<	< o m d s	<	<	< o m d s	<
"after" >	>	-	>	> oi mi d f	>	> oi mi d f	>	> oi mi d f	>	> oi mi d f	>	>	>
"during" d	d	<	>	d	-	< o m d s	> oi mi d f	<	>	d	> oi mi d f	d	< o m d s
"contains" di	di	< o m di fi	> oi di mi si	o oi d di =	di	o di fi	o di si	o di fi	oi di si	di fi o	di	di si oi	di
"overlaps" o	o	<	> oi di mi si	o d s	< o m di fi	< o m	o oi d di =	<	oi di si	o	di fi o	d s o	< o m
"overlapped by" oi	oi	< o m di fi	>	oi d f	> oi mi di si	o oi d di =	> oi mi	o di fi	>	oi d f	oi > mi	oi	oi di si
"meets" m	m	<	> oi mi di si	o d s	<	<	o d s	<	f fi =	m	m	d s o	<
"met by" mi		< o m di fi	>	oi d f	>	oi d f	>	s si =	>	d f oi	>	mi	mi
"starts" s	s	<	>	d	< o m di fi	< o m	oi d f	<	mi	s	s si =	d	< m o
"started by" si	si	< o m di fi	>	oi d f	di	o di fi	oi	o di fi	mi	s si =	si	oi	di
"finishes" f	f	<	>	d	> oi mi di si	o d s	> oi mi	m	>	d	> oi mi	f	f fi =
"finished by" fi	fi	<	> oi mi di si	o d s	di	o	oi di si	m	si oi di	o	di	f fi =	fi

Figure 5. The transitivity table for Allen's temporal relations.

The results stored in the table are complex terms, each consisting of from one to five disjuncts (average, 3.8). As computations progress over a chain of related intervals (A r1 B, B r2 C, C r3 D, ...), intermediate and resultant relations may contain as many as twelve terms (there are thirteen possible relationships, and the presence of all thirteen would indicate no knowledge at all). Furthermore, each computation may involve as many as 144 accesses to the transitivity table, though clever programming could substantially reduce this number.

By comparison, using endpoint relations, the cells of the corresponding transitivity table are either empty (indicating no known relationship) or contain exactly one relation (average, 0.4). See Figure 6. Hence, over a comparable chain of related intervals, each intermediate relation and the final relation will contain only a single term, and only one access to the transitivity table is required for the succeeding computation. This represents a potential gain in efficiency of from one to two orders of magnitude.

	saf	faf	sas	fas	fws	fwf	sws	swf	sbf	fbf	sbs	fbs
saf	saf	saf	sas	sas	sas	saf	sas	saf	-	-	-	-
faf	faf	faf	fas	fas	fas	faf	fas	faf	-	-	-	-
sas	saf	-	sas	-	-	-	sas	saf	-	-	-	-
fas	faf	-	fas	-	-	-	fas	faf	-	-	-	-
fws	faf	-	fas	-	fbs	fbf	fws	fwf	fbf	fbf	fbs	fbs
fwf	faf	faf	fas	fas	fws	fwf	fas	faf	-	fbf	-	fbs
sws	saf	-	sas	-	sbs	sbf	sws	swf	sbf	sbf	sbs	sbs
swf	saf	saf	sas	sas	sws	swf	sas	saf	-	sbf	-	sbs
sbf	-	-	-	-	sbs	sbf	-	-	-	sbf	-	sbs
fbf	-	-	-	-	fbs	fbf	-	-	-	fbf	-	fbs
sbs	-	-	-	-	sbs	sbf	sbs	sbf	sbf	sbf	sbs	sbs
fbs	-	-	-	-	fbs	fbf	fbs	fbf	fbf	fbf	fbs	fbs

Figure 6. Transitivity table for the endpoint relations.

Combining relations.

Certain relations between intervals imply other relations between those same intervals. For example, if X finishes before Y starts (X fbs Y), then clearly X starts before Y starts (X sbs Y). Figure 7 shows these implications.

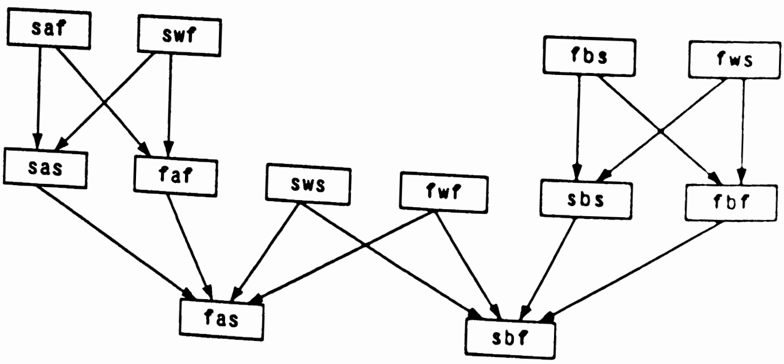


Figure 7. Certain relations imply others.

Two or more relationships may hold between a pair of intervals X and Y. If one relationship is implied by another, it may be omitted; for example, (X fbs Y) and (X sbs Y) may be simplified to (X fbs Y), since (X fbs Y) implies (X sbs Y). Other relationships may be compatible and add information, as for example (X sas Y) and (X fbf Y), which is equivalent to Allen's (X d Y). Certain relationships may be incompatible, as in (X saf Y) and (X fbs Y).

The table in Figure 8 shows how to combine two relations between the same pair of intervals. An X indicates that the relationships are incompatible, and any other entry indicates the result of conjoining them. For compatible entries, the result may of course always be represented as the conjunction of the row and column relationships; if this is the simplest representation, an ampersand (&) is shown, but if a more descriptive relationship applies (in either this system or Allen's system), it is shown. For example: (X saf Y) and (X fws Y) are incompatible; (X sas Y) and (X fbf Y) is equivalent to (X d Y).

&	saf	faf	sas	fas	fws	fwf	sws	swf	sbf	fbf	sbs	fbs
saf	saf	saf	saf	saf	X	X	X	X	X	X	X	X
faf	saf	faf	&	faf	X	X	si	swf	&	X	di	X
sas	saf	&	sas	sas	X	f	X	swf	&	d	X	X
fas	saf	faf	sas	fas	X	fwf	sws	swf	&	&	&	X
fws	X	X	X	X	fws	X	X	X	fws	fws	fws	X
fwf	X	X	f	fwf	X	fwf	=	X	fwf	X	fi	X
sws	X	si	X	sws	X	=	sws	X	sws	s	X	X
swf	X	swf	swf	swf	X	X	X	swf	X	X	X	X
sbf	X	&	&	&	fws	fwf	sws	X	sbf	fbf	sbs	fbs
fbf	X	X	d	&	fws	X	s	X	fbf	fbf	&	fbs
sbs	X	di	X	&	fws	fi	X	X	sbs	&	sbs	fbs
fbs	X	X	X	X	X	X	X	X	fbs	fbs	fbs	fbs

Figure 8. Combining relations between the same pair of intervals.
X = incompatible, & = no simpler representation for the conjunction.

Variations.

There is a simple extension of this system, and also a simple contraction. The contraction can be achieved by discarding the four relations fws, fwf, sws, and swf; the remaining relations are closed under transitivity. The philosophical justification would be that no two events (such as the start of one interval and the finish of another) can occur exactly at the same time, hence equality of time instants would be disallowed. This subsystem might well be adequate for some applications.

The extension would be achieved most simply by adding the negation of the existing relations. For example, (X ~saf Y) would mean that interval X does not start after interval Y finishes, that is, ((X sbf Y) or (X swf Y)). This is closely analogous to taking a mathematical system having the relations {<, =, >} and augmenting it with the relations {<=, ~=, >=}. The advantage is that these added relations may be represented directly, without the use of disjunctions (just as in arithmetic), and indeed other relationships between intervals will be simplified accordingly; the disadvantage is that the number of relations doubles, and the transitivity and compatibility tables each quadruple in size.

Conclusions.

The endpoint relations defined in this paper constitute a reformulation of Allen's temporal intervals. There is a simple translation from one system into the other, so in an important sense the two systems are equivalent. It is expected that theoretical work done using one set of relations would translate easily to the other set.

The system proposed in this paper has several advantages over Allen's original system:

- The new system has the property that longer symbolic expressions represent more knowledge, rather than less.

- Any knowledge of the relationship between two intervals can be captured in a conjunction of at most three relations, rather than a disjunction of at most eleven.

- The transitivity table is significantly more concise, and does not contain complex expressions.

- Disjunctions are largely replaced by conjunctions. To the extent that humans have difficulty with disjunctions, this may lead to a simpler user interface.

- The system is more amenable to the use of efficient time-point computations in special cases.

The two systems of relations between temporal intervals are quite compatible. For instance, it would be quite feasible to provide the user of an application with both kinds of relations, and to translate into one or the other of the systems for internal storage and manipulation of temporal information.

Bibliography.

[Allen 83] James F. Allen, *Maintaining Knowledge about Temporal Intervals*, Communications of the ACM 26(11), 1983.

[Allen 84] James F. Allen, *Towards a General Theory of Action and Time.* Artificial Intelligence 23(1984)123-154.

[Allen 85] James F. Allen and Patrick J. Hayes, *A Common-Sense Theory of Time.* Proceedings of IJCAI-85, 528-531.

[Matuszek 78] David Matuszek, *Disjunctive Complexity of Assertions as a Model of Perceived Complexity of Programs*, doctoral dissertation, The University of Texas, 1978.

[Neimark 70] E. D. Neimark and N. S. Slotnick, *Development of the Understanding of Logical Connectives*, Journal of Educational Psychology 61, 6 (1970), 451-460.

[Neisser 62] U. Neisser and P. Weene, *Hierarchies in Concept Attainment*, Journal of Experimental Psychology 64, 6 (1962), 640-645.

[Nitta 66] N. Nitta and S. Nagano, *Basic Logical Operations and their Verbal Expressions*, Research Bulletin of the National Institute for Educational Research, 1966, No. 7.

[Salatas 74] H. Salatas and L. E. Bourne, Jr. *Learning Conceptual Rules III: Processes Contributing to Rule Difficulty.* Memory and Cognition 2, 3 91974), 549-553.

[Tsang 87] Edward P. K. Tsang, *The Consistent Labeling Problem in Temporal Reasoning*, Proceedings of AAAI-87, 251-255.

[Valdes-Perez 87] Raul E. Valdes-Perez, *The Satisfiability of Temporal Constraint Networks*, Proceedings of AAAI-87, 256-260.

[Vilain 86] Marc Vilain and Henry Kautz, *Constraint Propagation Algorithms for Temporal Reasoning*, Proceedings of AAAI-86, 377-382.